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A FINITE STRAIN AXIALLY-SYMMETRIC SOLUTION FOR ELASTIC TUBES

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Abstract—An exact solution is given for the finite straining of elastic tubes under the action of internal and external pressure at the boundaries. The material model is described by Cauchy-elastic constitutive relations for isotropic Hookean-like materials. The governing equations are reduced to quadratures which provide a simple solution for the problem. Numerical results reveal the existence of a bursting pressure. Some further results are given for incompressible solids. Also discussed is the problem of a pressurized cylindrical cavity embedded in an infinite medium.

INTRODUCTION

Large deformation analysis of pressurized elastic tubes is a standard topic in finite elasticity[1]. Exact solutions though are still rare, depending on the details of the constitutive relations.

Here a simple solution is presented for that problem using a special model of a Cauchy-elastic solid. The true stresses are related to the strains by the classical Hookean relations, but the strain components are taken as the finite logarithmic measures. The same model has been employed in Ref. [2] for the analogous spherical shell model, and in Refs [3, 4] in the context of finite plasticity analysis.

A judicious choice of a new variable enables a direct solution, by quadratures, of the governing equations. Sample numerical computations show that the internal pressure (nondimensionalized with respect to the shear modulus) reaches a maximum point which—for tubes of moderate thickness—is almost independent of Poisson's ratio. That observation is supported by an approximate solution for thin-walled tubes. Some further simplifications are made for incompressible materials.

Finally the pressurized cavity, embedded in an infinite medium, is considered. It is shown that the applied pressure reaches an asymptotic value which is of the order of the elastic modulus.

GOVERNING EQUATIONS

The particular material model employed here is that of a Cauchy-elastic isotropic solid the constitutive relations of which are

$$\sigma_i = 2G \ln a_i + \lambda \ln J, \quad i = 1, 2, 3 \tag{1}$$

where σ_i are the principal Cauchy stress components, a_i the principal stretches, and J the volume ratio defined by

$$J = a_1 a_2 a_3. \tag{2}$$

The elastic constants G, λ are expressed by the usual Hookean connections

$$G = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{3}$$

where E is the elastic modulus and ν Poisson's ratio.

Consider now a thick-walled tube, deforming in a plane-strain axially-symmetric pattern, under the action of internal or external uniform normal stresses. Let r stand for the Lagrangean radial coordinate with (a, b) the undeformed internal and external radii, respectively. Introducing the non-dimensional coordinate

$$\rho = \frac{r}{a} \tag{4}$$

the principal logarithmic strains can be written as

$$\varepsilon_r = \ln(1+u'),\ \varepsilon_\theta = \ln\left(1+\frac{u}{\rho}\right) \tag{5}$$

where u is the radial displacement, nondimensionalized with respect to a, and the prime denotes differentiation with respect to ρ.

The constitutive relations (1) can now be put in the form

$$\sigma_r = 2G\ln(1+u')+\lambda\ln J \tag{6}$$

$$\sigma_\theta = 2G\ln\left(1+\frac{u}{\rho}\right)+\lambda\ln J \tag{7}$$

$$\sigma_z = \lambda\ln J \tag{8}$$

where $(\sigma_r, \sigma_\theta, \sigma_z)$ are the usual cylindrical polar stress components, and

$$J = (1+u')\left(1+\frac{u}{\rho}\right). \tag{9}$$

Equations (6)–(9) are compatible with the plane-strain constraint, but there should be no difficulty in extending the subsequent analysis to combinations of radial boundary stresses and a uniformly imposed axial strain.

Turning to the equilibrium requirements one has just the single radial equation, written in the undeformed reference configuration

$$\rho\sigma_r'+\frac{1+u'}{1+u/\rho}(\sigma_r-\sigma_\theta) = 0. \tag{10}$$

This equation is supplemented by the boundary data

$$\sigma_r(\rho = 1) = -p,\ \sigma_r(\rho = \eta) = t \tag{11}$$

where η is the initial radii ratio

$$\eta = \frac{b}{a}. \tag{12}$$

Since the volume ratio J is expressed by eqn (9) one can regard the three equations, eqns (6), (7) and (10), with the three unknowns $(\sigma_r, \sigma_\theta, u)$, as the governing system. It will be shown that this system admits a simple solution by quadratures.

SOLUTION

The solution centres on the new variable X defined by

$$X = \frac{1+u'}{1+u/\rho} = \rho \ln' (\rho+u). \tag{13}$$

Now, subtracting eqn (7) from eqn (6) gives

$$\sigma_r - \sigma_\theta = 2G \ln X \tag{14}$$

and the equilibrium equation, eqn (10), becomes

$$\rho\sigma_r' + 2GX \ln X = 0. \tag{15}$$

Combining eqns (6) and (9) one obtains

$$\sigma_r = (2G+\lambda) \ln (1+u') + \lambda \ln \left(1 + \frac{u}{\rho}\right) \tag{16}$$

or, with the aid of eqn (13)

$$\sigma_r = 2(G+\lambda) \ln \left(1 + \frac{u}{\rho}\right) + (2G+\lambda) \ln X. \tag{17}$$

The radial derivative of eqn (17) is

$$\sigma_r' = 2(G+\lambda) \ln' \left(1 + \frac{u}{\rho}\right) + (2G+\lambda) \frac{X'}{X}. \tag{18}$$

Observing now the identity, from eqn (13)

$$\ln' \left(1 + \frac{u}{\rho}\right) = \frac{1}{\rho}(X-1) \tag{19}$$

eqn (18) can be written as

$$\sigma_r' = 2(G+\lambda) \frac{X-1}{\rho} + (2G+\lambda) \frac{X'}{X}. \tag{20}$$

Inserting eqn (20) in eqn (15) results in the differential relation

$$\frac{d\rho}{\rho} = g(X)\, dX \tag{21}$$

where

$$g(X) = -\frac{1}{\alpha(X^2-X)+\beta X^2 \ln X} \tag{22}$$

and

$$\alpha = \frac{2G+2\lambda}{2G+\lambda} = \frac{1}{1-\nu}, \quad \beta = \frac{2G}{2G+\lambda} = \frac{1-2\nu}{1-\nu}. \tag{23}$$

Combining eqns (21) and (15) gives another differential relation, namely

$$\mathrm{d}\sigma_r = 2Gf(X)\,\mathrm{d}X \tag{24}$$

where

$$f(X) = -g(X)X\ln X. \tag{25}$$

The solution of the problem is provided by the integrals of eqns (21) and (24). Denoting by X_a and X_b the boundary values of variable X, one has

$$\ln\eta = \int_{X_a}^{X_b} g(X)\,\mathrm{d}X \tag{26}$$

$$t+p = 2G\int_{X_a}^{X_b} f(X)\,\mathrm{d}X. \tag{27}$$

These equations can be solved (using the method employed in Ref. [2]) without difficulty, for any given material constants, radii ratio and external loads. Once the values of X_a, X_b have been determined it is possible to get the radial stress, from eqn (24), as

$$\sigma_r = -p+2G\int_{X_a}^{X} f(\xi)\,\mathrm{d}\xi. \tag{28}$$

The circumferential stress follows from eqn (14), while the axial stress is obtained from the usual plain-strain relation

$$\sigma_z = \nu(\sigma_r+\sigma_\theta). \tag{29}$$

To find the radial displacement u one can rewrite eqn (17) in the form

$$\ln\left(1+\frac{u}{\rho}\right) = (1-2\nu)\frac{\sigma_r}{2G} - (1-\nu)\ln X \tag{30}$$

and use the expression for σ_r from eqn (28). The transformation from variable X to the Lagrangean coordinate ρ is given by the integral of eqn (21), namely

$$\rho = \exp\int_{X_a}^{X} g(\xi)\,\mathrm{d}\xi. \tag{31}$$

The fundamental functions $f(X)$ and $g(X)$ are shown in Fig. 1 for $X > 0$. Function $f(X)$ has an asymptote at $X = 0$ and its behaviour for small values of X is given essentially by

$$f(X) \sim -\frac{1}{\alpha}\ln X. \tag{32}$$

The asymptote $X = 0$ is approached by function $g(X)$ as well with the leading term

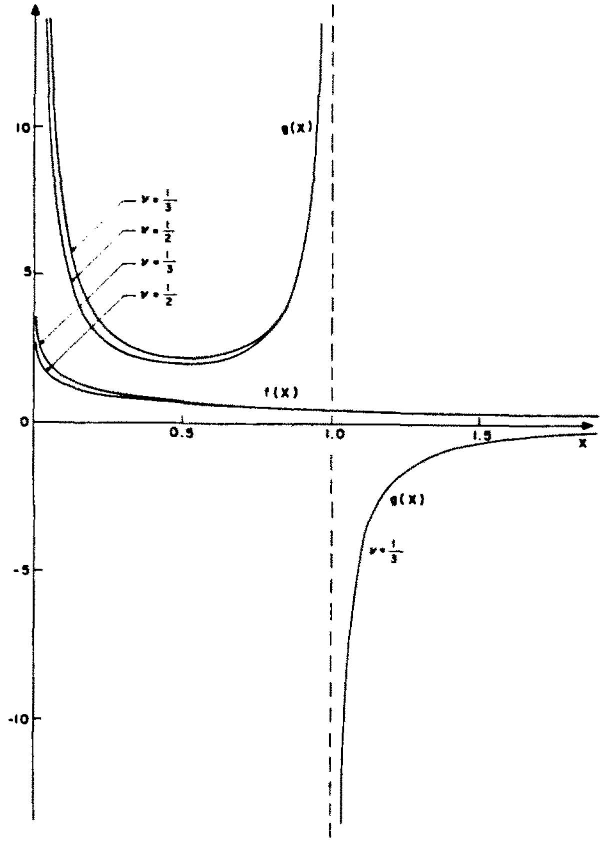

Fig. 1. The fundamental functions $f(X)$ and $g(X)$.

$$g(X) \sim \frac{1}{\alpha X}. \tag{33}$$

Function $g(X)$ however has a second asymptote at $X = 1$, and it can easily be verified via eqn (22) that, in the neighbourhood of $X = 1$

$$g(X) \sim \frac{1}{2(1-X)}. \tag{34}$$

These expansions will be used in the next section for investigating certain aspects of the solution. Note also (Fig. 1) that the influence of Poisson's ratio ν is appreciable only for low values of X.

DISCUSSION

Equations (26) and (27) have been solved for the case of internal pressure only ($t = 0$) using a standard numerical procedure. The calculations covered a wide range of radii ratios and different values of ν. The boundary values of variable X were found always in the range $0 < X_a < X_b < 1$. Tracing the load history of the tube revealed the expected existence of a maximum pressure, p_{max}, known also as the bursting pressure. The precise location of the maximum pressure point along the deformation path is obtained from the rate form of eqns (26) and (27), namely

$$g(X_b)\dot{X}_b - g(X_a)\dot{X}_a = 0 \tag{35}$$

$$f(X_b)\dot{X}_b - f(X_a)\dot{X}_a = \frac{\dot{p}}{2G} \tag{36}$$

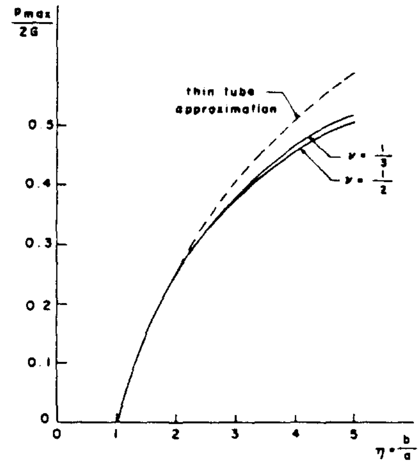

Fig. 2. Variation of the maximum pressure with radii ratio.

where the superposed dot denotes differentiation with respect to a time-like parameter. At the point of instability one has $\dot{p} = 0$ and eqns (35) and (36) may be conjoined, with the aid of eqn (25), to give

$$X_a^{X_a} = X_b^{X_b}. \tag{37}$$

The solution of eqns (37) and (26) yields the boundary values of variable X that correspond to p_{max}. Representative results for the maximum pressure are shown in Fig. 2. It is seen that Poisson's ratio has a small effect, on the non-dimensional bursting pressure, even for relatively thick tubes.

A further simplification of eqns (26) and (27) is possible for incompressible materials with $\nu = 1/2$. From eqns (23) it can be found that $\alpha = 2$, $\beta = 0$, and function $g(X)$ from eqn (22) becomes

$$g(X) = \frac{1}{2(X-X^2)}. \tag{38}$$

Equation (26) has now the exact integral

$$\eta^2 = \frac{X_a^{-1}-1}{X_b^{-1}-1} \tag{39}$$

and the internal pressure p, from eqn (27), is likewise

$$p = 2G\int_{X_a}^{X_b} \frac{\ln X}{2(X-1)}\,dX \tag{40}$$

where $G = E/3$. Of course, the calculation of the maximum pressure is now much simpler since eqns (39) and (37) can be combined to a single equation with one unknown.

When the tube is thin-walled eqns (26) and (27) may be replaced by the first-order approximations

$$\ln\eta = g(X_0)(X_b - X_a) \tag{41}$$

$$\frac{P}{2G} = f(X_0)(X_b - X_a) \tag{42}$$

where $2X_0 = X_a + X_b$. Thus, with the aid of eqn (25)

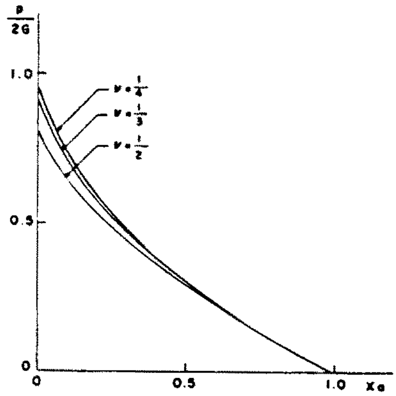

Fig. 3. Variation of cavity pressure with X_a. The asymptotic pressure is reached at $X_a = 0$.

$$\frac{P}{2G} = -(\ln\eta)X_0 \ln X_0. \tag{43}$$

This expression has a maximum at $X_0 = e^{-1}$ given by

$$\frac{P_{max}}{2G} = \frac{\ln\eta}{e}. \tag{44}$$

Note that within this approximation, the non-dimensional bursting pressure is independent of ν. Results obtained from eqn (44) are shown in Fig. 2 confirming the validity of thin tube approximation up to about $\eta = 2.5$.

At the other extreme one has the problem of a pressurized cavity embedded in an infinite medium. Here $\eta \to \infty$ and, in view of expression (34), condition (26) is satisfied with any $0 < X_a < 1$ provided that $X_b = 1$. The internal pressure, eqn (27), is now expressed as

$$\frac{p}{2G} = \int_{X_a}^{1} f(X)\,dX \tag{45}$$

with X_a decreasing from 1, at zero load, to 0. Figure 3 shows the dependence of the non-dimensional cavity pressure, eqn (45), on the value of X_a, for different values of ν. At the limit of $X_a \to 0$ the pressure reaches an asymptotic value which is equal to the area under the curve $f(X)$ in Fig. 1 over the range $0 \leqslant X \leqslant 1$. It is worth mentioning in this context that singularity (32) has a bounded integral in the vicinity of $X = 0$. Note, however, that the displacement at the cavity, eqn (30), becomes unbounded as the asymptotic pressure is approached.

For incompressible materials one has the exact result, for the asymptotic value of p

$$\frac{p_{asympt}}{2G} = \int_0^1 \frac{\ln X}{2(X-1)}\,dX = \frac{\pi^2}{12} \approx 0.8225 \tag{46}$$

or, put differently

$$p_{asympt} = \frac{\pi^2}{18}E \approx 0.5483E. \tag{47}$$

That result may be compared to the necking stress in uniaxial tension of material (1), with $\nu = 1/2$, given by $\sigma_{neck} = E$, and to the asymptotic pressure in a spherical cavity, embedded in an infinite medium, given by

$$p_{\text{asympt}} = \frac{2\pi^2}{27} E \approx 0.7311 E.$$

Material compressibility increases the magnitude of the asymptotic pressure sustained by the medium. With $\nu = 1/3$ one obtains $p_{\text{asympt}} \approx 0.702E$, while for $\nu = 1/4$, $p_{\text{asympt}} \approx 0.788E$ which is considerably above eqn (47).

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